The Commons Game[1]

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Abstract

In this note we present a complete characterization of the equilibria of a game modeling the "Tragedy of the commons" externality.

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Introduction

Since the appearance of Hardin's seminal paper (Hardin, 1968) on the tragedy of the commons several mathematical arguments in support of the ideas of the author have been published. Some of the arguments found their way into microeconomics textbooks as examples (Varian, 1993, Section 31.6) or exercises (Mass--Collel, Whinston, and Green, 1995, Example 11.D.4). The phenomenon has a straightforward game theoretical formulation an early version of which can be found in (Muhsam, 1973). It is thus rather surprising that the game model has never been fully exploited, although its conclusions are transparent and can be verified in a short and elementary way.

In this short note we present a complete mathematical analysis of a natural game theoretical model of the problem.

The setting of the problem is standard: We denote by $f(x)$ the value of the total yield of a herd of size x grazing on a common pasture. We assume as usual that $f:[0,\infty)\to[0,\infty)$ is twice differentiable, increasing, strictly concave and vanishes at 0. Further, we denote by a the cost of maintaining a unit of the herd (say, a cow), and consider x as a continuous variable. Under these assumptions the total profit π from grazing a herd on the pasture is

$$\pi(x) = f(x) - ax$$

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We assume

$$f'(0) > a,\ f'(\infty) < a \tag{1}$$

Although this hypothesis is usually not explicitly formulated, it is implicitly assumed and reasonable: the first inequality means that grazing is profitable if the size of the herd is sufficiently small whereas the second one infers that grazing ceases to be profitable if the herd is too large. Under assumption (1), maximum of the total profit is achieved at a unique $x = X$, which solves the equation

$$f'(X) = a \tag{2}$$

Assume now that there are N owners grazing their herds on the pasture. Then, we can treat the situation as a game with individual owners as players, sizes of their herds as strategies and their profits as their payoffs. As in the Cournot oligopoly model, a rational owner will seek to maximize his profit given the strategies – herd sizes – of the remaining owners.

The equilibrium of this model is of course the Nash one. It is our goal to determine all such equilibria and characterize their dependence on the number of the owners N. The result is summarized in the following

Theorem

For every N there is a unique Nash equilibrium. In this equilibrium all owners maintain herds of size Y/N, where $Y = Y(N)$ is the unique solution of the equation

$$(1-1/N)\,f(Y)/Y + f'(Y)/N = a \tag{3}$$

Y is strictly increasing with N, $Y \to X$ for $N \to 1$, $Y \to Z$ for $N \to \infty$, Z the unique solution of $f(Z)/Z = a$

This theorem exhibits in a transparent way that if there is more than one owner, the pasture will be overgrazed,[2] overgrazing increasing with the number of owners. Furthermore, with $N \to \infty$ the profits of the individual owners tend to zero and the formula for Y turns to that in (Varian, 1993). This limit we can interpret as an equilibrium under unlimited free entry of owners.

Proof of the Theorem

The proof is based on the inequality

$$f(Y)/Y > f'(Y) \text{ for all } Y > 0 \tag{4}$$

This is a well known and geometrically obvious property of concave functions but, for the convenience of the reader, we include its short analytic proof.

[2] That is, the total profit of the owners will fall short of the optimal one.

The function $\Phi(Y) = f(Y) - Yf'(Y)$ satisfies $\Phi(0) = 0, \Phi'(Y) = -Yf''(Y) > 0$ for $Y > 0$. Consequently, $\Phi(Y) > 0$ for $Y > 0$ which is equivalent to (4).

This inequality has the following consequence:

For all $0 \le q \le 1$, the equation

$$(1-q) f(Y)/Y + qf'(Y) = a \tag{5}$$

has a unique solution; this solution decreases with *q*.

To prove the claim, for $0 \le q \le 1$ denote $G(Y,q) = (1-q) f(Y)/Y + qf'(Y)$ for $Y > 0$, $G(0,q) = f'(0)$. Obviously, G is continuous on $[0,\infty) \times [0,1]$ and, by (1), $G(0,q) > a$. Existence and uniqueness of the solution of (5) will be proved once we show that for every q, G is decreasing in Y and $G(Y,q) < a$ for Y sufficiently large.

To simplify notation, drop the dependence on the variable *q*. We have

$$G'(Y) = \frac{1-q}{Y}\left[f'(Y) - \frac{f(Y)}{Y} \right] + qf''(Y)$$

By (4), the square bracket is negative and so is $f''(Y)$. Consequently, $G'(Y) < 0$.

By (1) there is a Y_0 such that $f'(Y) < a - \varepsilon$ for some $\varepsilon > 0$ and all $Y > Y_0$. For $Y > Y_0$ we have

$$\frac{f(Y)}{Y} = \frac{1}{Y}\int_0^{Y_0} f'(Y)dY + \frac{1}{Y}\int_{Y_0}^{Y} f'(Y)dY$$

$$< \frac{1}{Y}\int_0^{Y_0} f'(Y)dY + \frac{(a-\varepsilon)(Y-Y_0)}{Y}$$

The first term tends to 0 for $Y \to \infty$ while the second is smaller than $a - \varepsilon$. This completes the proof of existence and uniqueness of the solution of (5).

Let $Y = g(q)$ be the unique solution of the equation $G(Y,q) = a$; note that $g(0) = Z$ is the unique solution of $f(Z)/Z = a$.

To prove that g is decreasing we employ the implicit function theorem. Indeed, for $Y = g(q)$ we have

$$g'(q) = -\frac{\partial G(Y,q)/\partial q}{\partial G(Y,q)/\partial Y} = -\frac{-f(Y)/Y + f'(Y)}{(1-q)(f'(Y) - f(Y)/Y)Y + qf''(Y)} < 0$$

by (4) and the concavity of *f*.

Let now *Y* be the total size of the herd grazing on the pasture, Y_k the herd size of the *k*-th owner. Then, the profit π_k of the k – th owner will be

$$\pi_k(Y_k, Y) = \frac{Y_k}{Y} f(Y) - aY_k$$

the first term representing yield of the *k*-th owner obtained as the total yield multiplied by the share of his herd on the total grazing herd.

If $(Y_1,\ldots,Y_N)$ is a Nash equilibrium, $Y = Y_1 + \ldots + Y_N$, Y_k maximizes π_k for Y_j, $j \neq k$ fixed. The first order condition for maximum, $\frac{\partial}{\partial Y_k}\pi_k(Y_k, Y) = 0$, reads

$$(1 - \frac{Y_k}{Y})\frac{f(Y)}{Y} + \frac{Y_k}{Y} f'(Y) = a \tag{6}$$

or

$$Y_k(f'(Y) - f(Y)/Y) = aY - f(Y) \tag{7}$$

Due to (4) this equation has a unique solution for Y_k. Summing up (6) over k we obtain

$$(N-1) f(Y)/Y + f'(Y) = Na$$

Dividing by N we obtain (3). This is Equation (5) with $q = 1/N$. Thus, it has a unique solution $Y = Y(N)$. Being decreasing in q, this solution increases with *N*. Passing to the limit in (3) proves $Y(N) \to Z$ for $N \to \infty$. The values of Y_k in the Nash equilibrium have to be equal to the unique solution of (7) for $Y = Y(N)$ and sum up to *Y*(*N*), hence $Y_k = Y(N)/N$. This completes the proof.

References

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